Deriving the Orbit of the Earth Around the Sun

Problem Sheet

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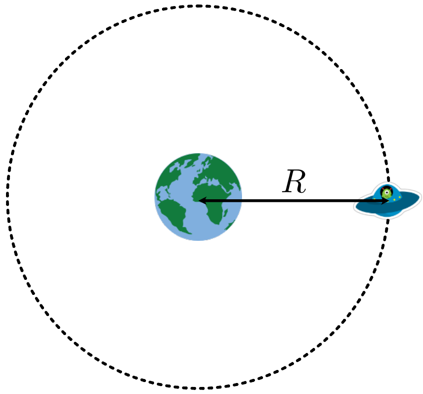
Solutions available at PhysicsWithElliot.com/orbits-mini



A flying saucer is orbiting the Earth!

(**a**) Suppose the alien-scientists aboard the saucer want to observe our planet from a constant distance R away. Noting that the gravitational force between the saucer and the planet will need to supply the centripetal force $\frac{mv^2}{R}$ that's always necessary to keep a mass m moving in circular motion with speed v, determine the time it takes for the saucer to make one revolution around the Earth. (Let the mass of the Earth be M and the saucer be m.)

(**b**) Determine the kinetic energy K, potential energy U, and the total energy E of the saucer. Show that they're related by $K = -\frac{1}{2}U$. (This is an example of the "virial theorem," which says how the average kinetic and potential energies of a particle are related. The relative coefficient will depend on the potential function in question.)

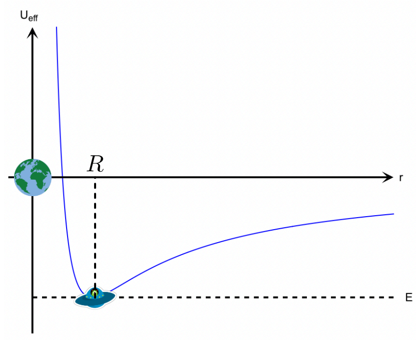
(**c**) Determine the angular momentum L of the saucer (with respect to the origin at the center of the Earth).

(**d**) Plug your results into the orbit equation,

$$r(\theta) = \frac{L^2}{km}\frac{1}{1+\epsilon\cos\theta},$$

where $k = GMm$ and $\epsilon = \sqrt{1 + \frac{2EL^2}{mk^2}}$, and check that it's consistent with what you expected.

(**e**)



Now forget about what you solved for in the previous parts. Given that the saucer is orbiting Earth in a circle of radius R, determine the energy E and angular momentum L of the saucer by finding the minimum of the effective potential $U_{\text{eff}}(r) = -\frac{GMm}{r} + \frac{L^2}{2mr^2}$. Hint: Take the slope of $U_{\text{eff}}(r)$ and set it equal to zero. Remember that the slope of r^n is nr^{n-1}.

The aliens briefly fire their booster rockets directly behind them, quickly increasing their speed by a factor $a > 1$, $v \to av$.

(**f**) Determine the saucer's new angular momentum and energy.

(**g**) How big can a be if the aliens want to remain in orbit around Earth and not go flying off into the solar system?

(**h**) Assuming the condition from the previous part is satisfied, so that the saucer remains in orbit, determine its new closest and farthest distances from the Earth, r_{min} and r_{max}, by finding the turning points in the effective potential.

(**i**) Write down the new orbit equation for $r(\theta)$ after the boosters have been fired. Check that it reproduces the minimum and maximum distances you found using the effective potential.