# Lagrangian and Hamiltonian Mechanics in Under 20 Minutes

Problem Sheet

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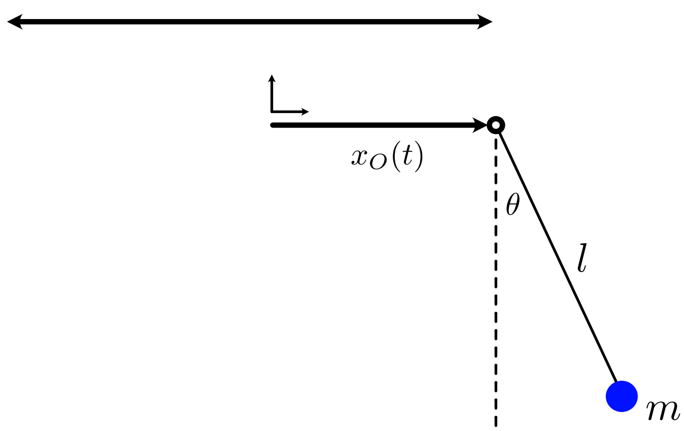
Solutions available at PhysicsWithElliot.com/lagrangian-hamiltonian-mini

One summer's night, you're strolling around a carnival that's been set up in your town for the weekend, when you pass by a booth with a sign that reads "Miss Madeline's Mind Mystifying Palace." Miss Madeline herself is standing next to it, and she explains that she is a hypnotist. She says that if you step in for a few minutes, she guarantees you'll be hypnotized into a calm and pleasant state of mind—or your money back. You're skeptical, but, figuring there's nothing to lose, you step inside through the curtain door.

Miss Madeline sits you down and dangles an old pocket watch in front of your eyes that she explains is a family heirloom with magical powers. She begins to steadily rock the chain back and forth. You start thinking to yourself how a dangling pocket watch is really, to good approximation, just a simple pendulum, when you suddenly feel yourself begin to enter a dreamlike state...




Setting the origin at the point where Miss Madeline is initially holding the end of the chain, suppose that she slides it back and forth, so that the position is $x_O(t)$ at time $t$. Suppose the chain has length $l$ and the watch itself has mass $m$, and treat the watch as a point particle and the chain as massless. Let $\theta$ be the angle that the watch chain makes with the vertical.

(**a**) Determine the coordinates $(x, y)$ of the particle in terms of $x_O$ and $\theta$.

(**b**) Determine the kinetic energy $K = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$ of the particle, the potential energy $U = mgy$, and the Lagrangian $L = K - U$. Express everything in terms of $\theta(t)$ and $x_O(t)$ (and their derivatives), and the given constants.

(**c**) Show that the Euler-Lagrange equation for $\theta$ is

$$\ddot{\theta} + \frac{g}{l}\sin\theta = -\frac{1}{l}\ddot{x}_O\cos\theta.$$

What are the momentum $p = \partial L/\partial\dot{\theta}$ and generalized force $\mathcal{F} = \partial L/\partial\theta$?

Suppose that Miss Madeline gently moves the end of the chain in a sinusoidal pattern: $x_O(t) = x_0 \sin(\Omega_0 t)$, where $x_0$ and $\Omega_0$ are constants. You find yourself feeling more and more at ease as you fall deeper into a trancelike state...

(**d**) Determine $\theta(t)$ for $t > 0$, assuming the initial conditions $\theta(0) = 0$ and $\dot{\theta}(0) = 0$. Assume that you can use the small angle approximation, which means that $\sin\theta \approx \theta$ and $\cos\theta \approx 1$ when $\theta$ is small. Also assume that $\Omega_0 \neq \Omega$, where $\Omega = \sqrt{g/l}$ is the natural frequency of the pendulum.

Hints: First write down what the general solution would have been when $x_O = 0$. Then guess a solution with $x_O$ turned back on of the form $\theta(t) = C\sin(\Omega_0 t)$, and plug it into the equation to figure out what $C$ must be. Finally, add these two solutions together, and apply the initial conditions.

Miss Madeline has gotten distracted talking to another carnival worker who has stopped by her booth, and she inadvertently begins to shake pendulum at frequency $\Omega_0 = \Omega$. Something dramatic happens, and you suddenly emerge from your trancelike state.

(**e**) In the previous part we had assumed that $\Omega_0 \neq \Omega$. But if they are equal, your guess $\theta(t) = C\sin(\Omega_0 t)$ from before won't work anymore. Try $\theta(t) = Dt\cos(\Omega_0 t)$ instead, and follow the same procedure to solve for $\theta(t)$. You'll find that by shaking the pendulum at its natural frequency, it will quickly oscillate up to large angles, and your solution using the small angle approximation will break down. This is an example of *resonance*.

Since Miss Madeline did succeed in hypnotizing your for a brief time, you decide not to ask for a refund, even though you were rudely awakened by the jostling pendulum when she shook it at resonance. As you walk back out into the carnival grounds, you reflect on how you might have understood the motion of her pocket watch using Hamiltonian mechanics.

(**f**) Determine the Hamiltonian for the system: write $p\dot{\theta} - L$, and then replace all the $\dot{\theta}$'s with the corresponding function of $p$, using the momentum you found in part (c). Be sure to check that when $x_O = 0$ you recover the original pendulum Hamiltonian $H = \frac{p^2}{2ml^2} - mgl\cos\theta$ from the video.

(**g**) Write Hamilton's equations, and show that they're equivalent to the Euler-Lagrange equation from part (c).